Regular Turán numbers

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Abstract

The regular Turán number of a graph F, denoted by $\mathrm{rex}(n, F)$, is the largest number of edges in a regular graph G of order n such that G does not contain subgraphs isomorphic to F. Giving a partial answer to a recent problem raised by Gerbner et al. [arXiv:1909.04980] we prove that $\mathrm{rex}(n, F)$ asymptotically equals the (classical) Turán number whenever the chromatic number of F is at least four; but it is substantially different for some 3-chromatic graphs F if n is odd.

1 Introduction

Let F be a fixed 'forbidden' graph. We denote

- $\mathrm{ex}(n, F)$ the maximum number of edges in a graph of order n that does not contain F as a subgraph — the classical *Turán number*;
- $\mathrm{rex}(n, F)$ the maximum number of edges in a *regular* graph of order n that does not contain F as a subgraph — the *regular Turán number*.

Of course $\mathrm{rex}(n, F) \leq \mathrm{ex}(n, F)$ holds for every F by definition.

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The Turán number of graphs is one of the most famous functions of graph theory, and it is well known to satisfy

$$\mathrm{ex}(n, K_{r+1}) = \binom{n}{2} - \sum_{i=0}^{r-1} \binom{\lfloor \frac{n+i}{r} \rfloor}{2} \quad \text{and} \quad \mathrm{ex}(n, F) = (1+o(1))\, \mathrm{ex}(n, K_{\chi(F)})$$

for all F with chromatic number $\chi(F) = r+1 \geq 3$, by the celebrated theorems of Turán[1] [11] and Erdős and Stone [7].

The regular Turán number was introduced recently by Gerbner, Patkós, Vizer, and the second author in [8], motivated by the study of singular Turán numbers introduced in [4]. It was proved in [8] that

- $\mathrm{rex}(n, K_3)$ is not a monotone function of n as $\mathrm{rex}(n, K_3) = \mathrm{ex}(n, K_3) = n^2/4$ if n is even, while

  $$\mathrm{rex}(n, K_3) \leq 2n^2/5$$

  if n is odd, the latter derived from a theorem of Andrásfai [2], which we shall present in details later as well as some recent progress motivated by this theorem;

- there exists a quadratic lower bound on $\mathrm{rex}(n, F)$ whenever $\chi(F) \geq 3$, namely

  $$\mathrm{rex}(n, F) \geq n^2/(g+6) - O(n)$$

  where g is the length of a shortest *odd* cycle in F (that is, the *odd girth* of F);

- if $\chi(F) = r+1 \geq 3$ and n is a multiple of r, then

  $$\mathrm{rex}(n, F) = (1+o(1))\, \mathrm{ex}(n, F)$$

  as $n \to \infty$, by the regularity of the Turán graph;

- if F is a tree on $p+1$ vertices and $\mathrm{ex}(n, F) \leq (p-1)n/2$, then $\mathrm{ex}(n, F) = \mathrm{rex}(n, F)$ for every n divisible by p.

[1] The unique extremal graph for K_{r+1} — the *Turán graph*, often denoted by $T_{n,r}$ — is obtained by partitioning the n vertices into r classes as equally as possible (each class has $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ vertices), and two vertices are adjacent if and only if they belong to distinct classes.

Problem 4.2 of [8] asks to determine $\liminf \operatorname{rex}(n,F)/n^2$ for non-bipartite graphs F. The goal of our present note is to solve this problem for a large class of graphs F, as expressed in the following results.

Theorem 1 *Let $r \geq 3$.*

(i) If n is a multiple of r, then $\operatorname{rex}(n,K_{r+1}) = \operatorname{ex}(n,K_{r+1})$.

(ii) If n is not a multiple of r, then $\operatorname{rex}(n,K_{r+1}) = \operatorname{ex}(n,K_{r+1}) - \Theta(n)$ as $n \to \infty$.

(iii) More exactly, if $n = qr + s$ with $1 \leq s \leq r-2$, and at least one of $r-s$ and q is even, then

$$\operatorname{rex}(n,K_{r+1}) = \operatorname{ex}(n,K_{r+1}) - \frac{(r-s)q}{2}.$$

(iv) If F is any graph with $\chi(F) \geq 4$, then $\operatorname{rex}(n,F) = (1-o(1))\operatorname{ex}(n,F)$.

(v) For $n = 3k+s$ with $s = 0,1,2$ we have $\operatorname{rex}(n,K_4) = kn = n \cdot \lfloor \frac{n}{3} \rfloor$.

Theorem 2 *Let F be a 3-chromatic graph.*

(i) If n is even, then $\operatorname{rex}(n,F) = (1-o(1))\operatorname{ex}(n,F) = n^2/4 + o(n^2)$; moreover, $\operatorname{rex}(n,K_3) = \operatorname{rex}(n,K_4 - e) = n^2/4$.

(ii) If n is odd, and $F = K_3$ or $F = K_4 - e$ or F is a unicyclic graph with C_3 as its cycle, then $\operatorname{rex}(n,F) = n^2/5 - O(n)$.

(iii) If $F = K_3$ and $n = 5k+s$ is odd, then $\operatorname{rex}(n,F) = kn = n \cdot \lfloor \frac{n}{5} \rfloor$.

Theorem 3 *If $\chi(F) = 3$, and F has odd girth g, then*

$$\operatorname{rex}(n,F) \geq n^2/(g+2) - O(n)$$

for every odd n. Moreover, if $F = C_5$ or F is a unicyclic graph with C_5 as its cycle, then the asymptotic equality

$$\operatorname{rex}(n,F) = n^2/7 - O(n)$$

is valid. More precisely, if $n = 7k+s$ is odd, and n is sufficiently large with respect to F, then $\operatorname{rex}(n,F) = kn = n \cdot \lfloor \frac{n}{7} \rfloor$.

It should be noted that almost nothing is known about the behavior of $\operatorname{rex}(n, F)$ in case of bipartite graphs F, apart from a couple of remarks in the concluding section of [8]. Moreover, the following problem remains widely open for 3-chromatic graphs.

Problem 1 *Determine* $\operatorname{rex}(n, F)$*, or its asymptotic growth as* $n \to \infty$*, for graphs* F *with* $\chi(F) = 3$ *for odd* n*.*

As a particular case, we expect that the exact results on K_3 and C_5 extend to a unified formula for every odd cycle.

Conjecture 1 *If* $g \geq 3$ *and* $n = k\,(g+2) + s$ *with* $0 \leq s \leq g+1$*, and both* g *and* n *are odd, then* $\operatorname{rex}(n, C_g) = kn = n \cdot \lfloor \frac{n}{g+2} \rfloor$*.*

The unicyclic extensions given in Theorems 2 and 3 are consequences of the following principle. It has an analogous implication also for those graphs whose unique non-trivial block is $K_4 - e$.

Proposition 1 *If the growth of* $\operatorname{rex}(n, F)$ *is superlinear in* n*, and* F^+ *is a graph obtained from* F *by inserting a pendant vertex, then* $\operatorname{rex}(n, F^+) = \operatorname{rex}(n, F)$ *for every sufficiently large* n*.*

At the end of this introduction let us recall the full statement of Andrásfai's theorem, which plays an essential role in the current context.

Theorem 4 ([2]) *If* G *is a triangle-free graph on* n *vertices and with minimum degree* $\delta(G) > 2n/5$*, then* G *is bipartite.*

This theorem is the source of motivation for recent research which is also related and relevant to our Theorem 3 and Conjecture 1; see [1, 6, 9, 10]. We explicitly quote the following very useful generalization, proved by Andrásfai, Erdős, and T. Sós, as read out from the combination of their Theorem 1.1 and Remark 1.6.

Theorem 5 ([3]) *For each odd integer* $k \geq 5$ *and each integer* $n \geq k$*, if* G *is a simple* n*-vertex graph with no odd cycles of length less than* k *and with minimum degree* $\delta(G) > 2n/k$*, then* G *is bipartite.*

2 Forbidden graphs with chromatic number at least 4

In this section we prove Theorem 1. We assume throughout that $n = qr + s$, where q is an integer and $0 \le s \le r - 1$ holds.

Proof of Part *(i)*

Whenever n is a multiple of r, the equality $\mathrm{rex}(n, K_{r+1}) = \mathrm{ex}(n, K_{r+1})$ is clear because the r-partite Turán graph $T_{n,r}$ is regular in all such cases.

Proof of Part *(ii)*

First we argue that $\mathrm{ex}(n, K_{r+1}) - \mathrm{rex}(n, K_{r+1})$ is at least a linear function of n if r does not divide n. We see from Turán's theorem that the degree of regularity cannot exceed $(1 - 1/r)\, n$ if the graph is K_{r+1}-free. For $n = qr + s$ with $0 < s < r$ it means that the degrees are at most $n - (n + r - s)/r$. This value is the degree of vertices in the s larger classes of the Turán graph; but the $r - s$ smaller classes consist of vertices of degree $n - (n + r - s)/r + 1$. Hence the degree sum in a regular graph is smaller by at least $\frac{r-s}{r}\, n - O(1)$.

Next, we construct r-chromatic (hence also K_{r+1}-free) regular graphs to show that the difference between $\mathrm{ex}(n, K_{r+1})$ and $\mathrm{rex}(n, K_{r+1})$ is at most $O(n)$. For $n = qr + s$ with $0 < s < r$ the Turán graph has s classes of size $q + 1$ and $r - s$ classes of size q. Putting this in another way, the vertices in s classes have degree $n - q - 1$, and in $r - s$ classes have degree $n - q$.

We are going to delete $O(n)$ edges and obtain a regular graph. For this purpose we shall use Dirac's theorem [5] as a lemma, which states that if the degree of every vertex in a graph H is at least half of the number of vertices, then H has a Hamiltonian cycle.

Assume first that both s and $r - s$ are at least 2. Since the degree sum is even, the total size of the odd-degree classes together is also even, and the subgraph induced by them is Hamiltonian. Hence the union of these classes admits a 1-factor, which we remove. If this is the larger degree, then we are done. Otherwise the subgraph induced by the larger degrees also has a Hamiltonian cycle, whose removal leaves a regular graph of degree $n - q - 2$.

Assume next that $s = 1$; i.e., only one class contains vertices of low degree. If the total size of the $r - 1$ high-degree classes is even, we remove a 1-factor from the subgraph induced by them, and we are done. On the other hand, if their total size is odd, their degree must be even. That is, both q

and $r-1$ are odd; in particular, $r-1 \geq 3$ holds. In this situation our plan is to delete two edges from each such vertex, and one edge from each vertex of the low-degree class.

We begin with the single class, which has odd size. We omit one edge from each of its vertices — mutually disjoint edges — in such a way that the other ends of those $q+1$ edges are distributed as equally as possible among the $r-1 \geq 3$ high-degree classes. Then Dirac's condition is valid for the high-degree ends of the omitted matching and also for the subgraph from which no edges have been omitted so far. Hence both parts are Hamiltonian. Since $q+1$ is even, we can omit a perfect matching from the former, and a Hamiltonian cycle from the latter, thus obtaining a regular graph.

Finally, consider the case of $s = r-1$; i.e., only one class contains vertices of high degree. We must remove edges from the high-degree class, which means that also some low-degree vertices will decrease their degree. Similarly to the previous case, here again, the parity of classes will matter.

If the high-degree class has even size q, we decrease its degrees by 3, distributing the neighbors equally among the low-degree classes, and inside the neighbors we delete a 1-factor. In the rest of low-degree vertices we delete a Hamiltonian cycle.

Suppose that the high-degree class has odd size q. Then the number of low-degree vertices is even because each such class has even size $q+1$. In particular, n is odd. We now delete two edges from each high-degree vertex, this decreases $2q$ of the low degrees by 1. From the other $n-3q$ (i.e., even number of) vertices we delete a 1-factor which exists since also this subgraph is Hamiltonian. This modification yields a regular graph, and completes the proof.

Proof of Part (*iii*)

Recall that the number of classes of high-degree vertices in the Turán graph is $r-s$, and these classes have cardinality q each. Note further that their union induces a Hamiltonian subgraph whenever $s \leq r-2$. Under the assumption that at least one of $r-s$ and q is even, the length $(r-s)\,q$ of a corresponding Hamiltonian cycle is even, hence contains a perfect matching, say M. Removing M from $T_{n,r}$ we obtain a K_{r+1}-free regular graph, and the degree is largest possible, according to the first part of the proof of (*ii*) as given above.

Proof of Part *(iv)*

The r-colorable construction given above for (ii) proves that $\mathrm{rex}(n, F)$ is at least $\mathrm{ex}(n, K_{r+1}) - O(n)$ if $\chi(F) = r + 1$. On the other hand, as mentioned already, $\mathrm{ex}(n, F) = (1 + o(1))\,\mathrm{ex}(n, K_{r+1})$ is valid by the Erdős–Stone theorem, implying that $\mathrm{rex}(n, F)$ cannot be larger.

Proof of Part *(v)*

The following list is a summary of optimal constructions according to $n \pmod 3$.

- $n = 3k$ — the complete 3-partite graph with equal classes, i.e. $T_{n,3}$, is regular.
- $n = 3k+1$ — here $T_{n,3}$ has vertex classes of respective sizes $k, k, k+1$; it can be made regular by removing a perfect matching between the two classes of size k.
- $n = 3k+2$ — here $T_{n,3}$ has vertex classes of respective sizes $k, k+1, k+1$; it can be made regular by removing a matching of k edges between the class of size k and each class of size $k+1$ (hence removing kP_3 from $T_{n,3}$), moreover deleting the edge that joins the two vertices whose degree has not been decreased by the removal of the two matchings. This is optimal because removing an edge from a vertex of high degree decreases the degree of a vertex of low degree, too.

3 3-chromatic forbidden graphs

Let us begin this section with the proof of Proposition 1, as it is applicable for Theorems 2 and 3 as well.

Proof of Proposition 1

Certainly we have $\mathrm{rex}(n, F^+) \geq \mathrm{rex}(n, F)$. Suppose that G is a regular graph of order n, which is extremal for F^+. If G is F-free, then the reverse inequality $\mathrm{rex}(n, F^+) \leq \mathrm{rex}(n, F)$ also holds and the assertion follows immediately. On the other hand, if $F \subset G$ but $F^+ \not\subset G$, the degree of regularity in G must be smaller than $|V(F^+)|$, for otherwise it would be possible to extend F to F^+ in G. This fact puts a $O(n)$ upper bound on $|E(G)|$, contradicting the superlinear growth of $\mathrm{rex}(n, F)$. Thus the largest F^+-free regular graphs are F-free, too, as n gets large.

Proof of Theorem 3, general lower bound

We construct a graph of order n and odd girth $g+2$, hence it will not contain F as a subgraph. Let us write n in the form $n = (g+2)\cdot a + 2b$ where b is an integer in the range $0 \le b \le g+1$. We start with a blow-up of $C_{g+2} = v_1 v_2 \dots v_{g+2}$ by substituting independent sets $A_1, A_2, \dots, A_{g+2}$ into its vertices, completely joining A_i with A_{i+1} for $i = 1, \dots, g+2$ (where $A_{g+3} := A_1$). We let $|A_1| = |A_2| = a+b$, and $|A_i| = a$ for all $3 \le i \le g+2$. The degree of a vertex v in this graph is $2a+b$ if $v \in A_1 \cup A_2 \cup A_3 \cup A_{g+2}$, and it is $2a$ otherwise. The graph is regular if $b = 0$, and it will be made regular by the removal of

$$2ab+b^2 = b\left(\frac{2n-4b}{g+2}+b\right) = \frac{b\cdot(2n+(g-2)b)}{g+2} \le 2n - \frac{2n-(g-2)(g+1)^2}{g+2}$$

edges otherwise.

Between A_{g+2} and A_1 we remove a bipartite graph H such that all vertices of H in A_{g+2} have degree b, and all degrees in A_1 are $\lfloor\frac{ab}{a+b}\rfloor$ or $\lceil\frac{ab}{a+b}\rceil$. Such H clearly exists. We also remove a bipartite graph isomorphic to H between A_2 and A_3, such that the degree-a vertices are in A_3.

If ab is a multiple of $a+b$, then the current vertex degrees in $A_1 \cup A_2$ are $2a+b-\frac{ab}{a+b} = 2a+\frac{b^2}{a+b}$. Then removing a regular bipartite graph of degree $\frac{b^2}{a+b}$, hence with b^2 edges, yields a $(2a)$-regular graph of order n.

Otherwise, if ab is not divisible by $a+b$, we first remove a perfect matching between the vertices of degree $2a+b-\lfloor\frac{ab}{a+b}\rfloor$ in $A_1 \cup A_2$. After that, the bipartite graph induced by $A_1 \cup A_2$ is regular, hence deleting a regular subgraph of degree $b-\lceil\frac{ab}{a+b}\rceil$ from it, we obtain a $(2a)$-regular graph of order n.

Proof of Theorem 3 for asymptotics of C_5 and unicyclic graphs

We have to prove that $\mathrm{rex}(n, F) \le n^2/7$, if n is not very small. By Proposition 1 it is enough to deal with the case of $F = C_5$. Let G be a C_5-free regular graph with $\mathrm{rex}(n, C_5)$ edges. If G is triangle-free, the proof is done by Theorem 5, because the odd girth cannot be exactly 5. On the other hand it was proved in [9, Lemma 33] that if G contains a triangle and the degrees are greater than $(1/6+\epsilon)\,n$, for any $\epsilon > 0$ and sufficiently large

n, then G also has a C_5. Hence in C_5-free graphs with triangles we cannot have more than $n^2/12 + o(n^2)$ edges.

We postpone the proof of the exact formula for $\mathrm{rex}(n, C_5)$ to the end of this paper, due to its similarity to the argument concerning $\mathrm{rex}(n, C_3)$.

Proof of Theorem 2, Parts (*i*) and (*ii*)

If F is 3-chromatic and n is even, the lower bound of $n^2/4$ is shown by the complete bipartite graph $K_{n/2,n/2}$, while an asymptotic upper bound follows by the Erdős–Stone theorem as

$$\mathrm{rex}(n, F) \leq \mathrm{ex}(n, F) = (1 - o(1))\, \mathrm{ex}(n, K_3) = n^2/4 + o(n^2).$$

The tight results $\mathrm{rex}(n, K_3) = \mathrm{rex}(n, K_4 - e) = n^2/4$ follow from the facts that the Turán number of K_3 and also of $K_4 - e$ is $n^2/4$.

Assume that n is odd. The lower bound $\mathrm{rex}(n, K_3) \geq n^2/5 - O(n)$ is a particular case of the previous construction, putting $g = 3$. Moreover, in triangle-free regular graphs we cannot have more than $n^2/5$ edges, due to Theorem 4 and by the fact that every regular bipartite graph has an even order. This already settles the case of K_3 (and also of the unicyclic graphs having a 3-cycle). For $K_4 - e$ assume that x, y, z induce K_3. If this triangle cannot be extended to $K_4 - e$, then the degree of x, y, z is at most $2 + (n - 3)/3 = n/3 + 1$, thus by the condition of regularity the number of edges is at most $n^2/6 + n/2$, which is much less than $n^2/5$ if n is large.

Proof of Theorem 2, Part (*iii*)

Let G be a triangle-free regular graph on n vertices, with $|E(G)| = \mathrm{rex}(n, K_3)$. Assume that $n = 5k + s$, where $s = 0, 1, 2, 3, 4$ and n is odd. As a consequence, $k + s$ is odd as well; however, the degree d of regularity must be even. From Theorem 4 we also know that $d \leq \lfloor 2n/5 \rfloor = 2k + \lfloor 2s/5 \rfloor$. A more careful look verifies that the last term $\lfloor 2s/5 \rfloor$ will vanish with respect to d, and in every possible case we have

$$d \leq 2k = 2(n - s)/5.$$

This is clear for $s = 0, 1, 2$ due to the floor function. But also for $s = 3, 4$ we have that $\lfloor 2s/5 \rfloor = 1$ is an odd number, whereas d must be even, hence d cannot exceed the largest even integer under $2k + 1$, which is actually $2k$. Thus, $|E(G)| \leq kn$.

It remains to show that for every odd $n = 5k + s$ there exists a K_3-free graph of order n which is $2k$-regular. The general principle of the construction is to substitute independent sets $A_1, \ldots, A_5$ into the vertices of C_5, where each edge of C_5 becomes a complete bipartite graph between the corresponding two sets A_i, A_{i+1} cyclically; and then delete some edges so that a regular graph is obtained. We are going to describe these constructions for each s one by one, specifying the sequences $(|A_1|, \ldots, |A_5|)$ as follows.

- $s = 0: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k, k, k, k, k)$

  This graph is $2k$-regular.

- $s = 1: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k + 1, k + 1, k, k - 1, k)$

  This graph becomes $2k$-regular after the deletion of a perfect matching from the induced subgraph $G[A_1 \cup A_2]$.

- $s = 2: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k + 1, k + 1, k, k, k)$

  This graph becomes $2k$-regular after the deletion of a matching of size k from $G[A_1 \cup A_5]$ and from $G[A_2 \cup A_3]$, moreover the edge between the two unmatched vertices of $A_1 \cup A_2$.

- $s = 3: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k + 1, k + 1, k + 1, k, k)$

  This graph becomes $2k$-regular after the deletion of a perfect matching from each of the induced subgraphs $G[A_1 \cup A_2]$, $G[A_2 \cup A_3]$, $G[A_4 \cup A_5]$.

- $s = 4: \quad (|A_1|, |A_2|, |A_3|, |A_4|, |A_5|) = (k + 2, k + 2, k, k, k)$

  This graph can be made $2k$-regular in the following way. Specify two vertices $a_1', a_1'' \in A_1$ and $a_2', a_2'' \in A_2$; set $A_1' = A_1 \setminus \{a_1', a_1''\}$ and $A_2' = A_2 \setminus \{a_2', a_2''\}$. Delete a 2-factor from $G[A_1' \cup A_5]$ and from $G[A_2' \cup A_3]$; and delete the edges of the 4-cycle $a_1' a_2' a_1'' a_2''$.

Proof of Theorem 3 for exact $\mathrm{rex}(n, C_5)$

Since the proof is very similar to that of the exact formula for $\mathrm{rex}(n, K_3)$, we give a more concise description here. Let now $n = 7k+s$, where $0 \le s \le 6$. We have already seen that the degree d of regularity satisfies $d \le \lfloor 2n/7 \rfloor = 2k + \lfloor 2s/7 \rfloor \le 2k + 1$; and d must be even, thus $d \le 2k$. It remains to give suitable substitutions of sets $A_1, \ldots, A_7$ into the vertices of C_7 in such a way that the graphs can be made $2k$-regular by the deletion of some edges. Below we define a sequence $|A_1|, |A_2|, |A_3|, |A_4|, |A_5|, |A_6|, |A_7|$ similar to the case of C_5, now for each $s = 0, 1, \dots, 6$.

- $s = 0: \quad k, k, k, k, k, k, k$

  Nothing to delete.

- $s = 1: \quad k + 1, k, k, k + 1, k, k - 1, k$

  Delete a 1-factor from $G[A_2 \cup A_3]$.

- $s = 2: \quad k + 1, k + 1, k, k, k, k, k$

  Delete a matching of size k from $G[A_1 \cup A_7]$ and from $G[A_2 \cup A_3]$, and the edge between the two unmatched vertices of $A_1 \cup A_2$.

- $s = 3: \quad k + 1, k + 1, k, k, k + 1, k, k$

  Delete a 1-factor from $G[A_1 \cup A_2]$, from $G[A_3 \cup A_4]$, and from $G[A_6 \cup A_7]$.

- $s = 4: \quad k + 2, k + 2, k, k, k, k, k$

  Delete the edges of a $H \cong C_4$ in $A_1 \cup A_2$, and a 2-factor from $G[(A_1 \cup A_7) \setminus V(H)]$ and from $G[(A_2 \cup A_3) \setminus V(H)]$.

- $s = 5: \quad k + 2, k + 2, k, k, k + 1, k, k$

  Delete a C_4 from $G[A_1 \cup A_2]$, and a matching of size k from each consecutive pair of A_i, A_{i+1} along the cycle (also including A_7, A_1 as cyclically consecutive), except $G[A_4 \cup A_5]$ and $G[A_5 \cup A_6]$.

- $s = 6: \quad k + 2, k + 2, k, k, k + 2, k, k$

  Delete a 2-factor from $G[A_1 \cup A_2]$, from $G[A_3 \cup A_4]$, and from $G[A_6 \cup A_7]$.

After the deletions, all graphs are $2k$-regular, completing the proof of the theorem.

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References

[1] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, R. Morris: The chromatic thresholds of graphs. Adv. Math. 235 (2013), 261–295.

[2] B. Andrásfai: Graphentheoretische Extremalprobleme. Acta Math. Acad. Sci. Hungar. 15 (1964), 413–418.

[3] B. Andrásfai, P. Erdős, V. T. Sós: On the connection between chromatic number, maximal clique and minimal degree of a graph. Discrete Mathematics 8 (1974), 205–218.

[4] Y. Caro, Zs. Tuza: Singular Ramsey and Turán numbers. Theory and Applications of Graphs 6 (2019), 1–32.

[5] G. A. Dirac: Some theorems on abstract graphs. Proceedings of the London Mathematical Society, 3rd Ser. 2 (1952), 69–81.

[6] O. Ebsen, M. Schacht: Homomorphism thresholds for odd cycles. arXiv:1712.07026

[7] P. Erdős, A. H. Stone: On the structure of linear graphs. Bulletin of the American Mathematical Society, 52 (1946), 1087–1091.

[8] D. Gerbner, B. Patkós, Zs. Tuza, M. Vizer: Singular Turán numbers and WORM-colorings. arXiv:1909.04980

[9] S. Letzter, R. Snyder: The homomorphism threshold of $\{C_3, C_5\}$-free graphs. Journal of Graph Theory, 90 (2019) 83–106.

[10] S. Messuti, M. Schacht: On the structure of graphs with given odd girth and large minimum degree. Journal of Graph Theory, 80 (2015) 69–81.

[11] P. Turán: Egy gráfelméleti szélsőértékfeladatról (On an extremal problem in graph theory). Matematikai és Fizikai Lapok, 48 (1941), 436–452. (In Hungarian)